

Algebra and functions 1

1.1 Simplifying rational expressions

A rational expression is a fraction. We simplify algebraic fractions using the same rules that we use when simplifying numerical fractions.

Example 1

Simplify

a $\frac{12}{15}$

b $\frac{7a}{5a^2}$

c $\frac{3x+9}{6x}$

d $\frac{x^2+2x}{x+2}$

e $\frac{x^2+2x-3}{x^2+5x+6}$

a $\frac{12}{15}$. Divide numerator and denominator by 3, we have $\frac{12}{15} = \frac{4}{5}$.

b $\frac{7a}{5a^2}$. Dividing numerator and denominator by a , we have $\frac{7a}{5a^2} = \frac{7}{5a}$.

c $\frac{3x+9}{6x}$. A common mistake is often made with fractions like this
Do **not** cancel the x s.

Instead we write $\frac{3x+9}{6x} = \frac{3(x+3)}{6x} = \frac{(x+3)}{2x}$.

d Factorising the numerator, $\frac{x^2+2x}{x+2} = \frac{x(x+2)}{x+2} = x$

e Factorising $\frac{x^2+2x-3}{x^2+5x+6} = \frac{(x-1)(x+3)}{(x+2)(x+3)} = \frac{x-1}{x+2}$

EXERCISE 1

Simplify as far as possible.

1 $\frac{24}{30}$

2 $\frac{4a^2}{a}$

3 $\frac{x}{3x}$

4 $\frac{8x^2}{2x^2}$

5 $\frac{2a}{4b}$

6 $\frac{6m}{2m}$

7 $\frac{5ab}{10b}$

8 $\frac{8ab^2}{4ab}$

9 $\frac{15y}{20y^2}$

10 $\frac{11xy}{12x^2}$

11 $\frac{8ya^2}{12a}$

12 $\frac{12m^2n^2}{3mn^3}$

13 Sort these into four pairs of equivalent expressions.

A $\frac{x^2}{4x}$
E $\frac{x+1}{x}$

B $\frac{x(x+1)}{x^2}$
F $\frac{x+2}{x}$

C $\frac{x^2+x}{x^2-x}$
G $\frac{x}{4}$

D $\frac{3x+6}{3x}$
H $\frac{x+1}{x-1}$

In questions 14 to 29 simplify as far as possible.

14 $\frac{7a^2b}{35ab^2}$

15 $\frac{(2a)^2}{4a}$

16 $\frac{7yx}{8xy}$

17 $\frac{3x}{4x-x^2}$

18 $\frac{5x+2x^2}{3x}$

19 $\frac{9x+3}{3x}$

20 $\frac{4a+5a^2}{5a}$

21 $\frac{5ab}{15a+10a^2}$

22 $\frac{3x-x^2}{2x}$

23 $\frac{12x+6}{6y}$

24 $\frac{3(2x^2+5x)}{6x}$

25 $\frac{xy+x^2y}{x}$

26 $\frac{5x^3+4x^2}{x(3x^2-2x)}$

27 $\frac{a+4a^2}{ab+ab^2}$

28 $\frac{4+8x+8x^2}{4x}$

29 $\frac{54mn^2-27m^2n}{18(mn)^2}$

30 Look at each of the following and decide whether or not the statement is 'true' or 'false'.

a $\frac{4x^2+8}{2x} = \frac{4x+8}{2}$

b $\frac{x^2y+xy^2}{xy} = x+y$

c $\frac{4ab^2c^3}{a^3b^2c} = \frac{4c^2}{a^2}$

d $\frac{5x+6}{6y} = \frac{5x+1}{y}$

31 a Factorise the expression x^2-x-6 .

b Hence simplify the expression $\frac{x^2-x-6}{(x+2)}$

32 Simplify as far as possible.

a $\frac{x^2-3x-4}{x+1}$

b $\frac{x^2+x-6}{x^2+2x-3}$

c $\frac{x^2+3x-10}{x^2-4}$

d $\frac{x^2-3x}{x^2-2x-3}$

e $\frac{x^2+4x}{2x^2-10x}$

f $\frac{x^2+6x+5}{x^2-x-2}$

g $\frac{2x^2+x}{4x^2-1}$

h $\frac{2x^2-5x-3}{2x^2-3x-9}$

i $\frac{3x^2+x-4}{9x^2-16}$

33 Write the expression $\frac{x+\frac{1}{2}}{x+\frac{1}{3}}$ in a more simple form without fractions on the numerator and denominator.

[Hint: Multiply numerator and denominator by 6].

34 Write the following in a more simple form without fractions.

a $\frac{x+\frac{1}{x}}{x}$

b $\frac{2x-\frac{1}{x}}{\frac{1}{x}}$

c $\frac{x-\frac{1}{2}}{\frac{1}{2}}$

$$d \frac{3x + \frac{1}{4}}{\frac{1}{4}}$$

$$e \frac{5x - \frac{1}{3}}{\frac{1}{6}}$$

$$f \frac{\frac{1}{4} - x}{\frac{1}{2}}$$

$$g \frac{3x + \frac{1}{x}}{x + \frac{2}{x}}$$

$$h \frac{x - \frac{4}{x}}{x - 2}$$

$$i \frac{2x - 3}{4x - \frac{9}{x}}$$

$$j \frac{1 - \frac{1}{x^2}}{\frac{1}{x^2}}$$

$$k \frac{1 - \frac{4}{x^2}}{1 - \frac{2}{x}}$$

$$l \frac{1 + \frac{2}{x} - \frac{3}{x^2}}{1 + \frac{3}{x} - \frac{4}{x^2}}$$

Addition and subtraction of algebraic fractions

Example 2

a Write as a single fraction $\frac{2}{x} + \frac{3}{y}$

The L.C.M. of x and y is xy .

$$\therefore \frac{2}{x} + \frac{3}{y} = \frac{2y}{xy} + \frac{3x}{xy} = \frac{2y + 3x}{xy}$$

b Write as a single fraction $\frac{4}{x} + \frac{5}{x-1}$

The L.C.M. of x and $(x-1)$ is $x(x-1)$

$$\begin{aligned} \therefore \frac{4}{x} + \frac{5}{x-1} &= \frac{4(x-1) + 5x}{x(x-1)} \\ &= \frac{9x - 4}{x(x-1)} \end{aligned}$$

EXERCISE 2

1 Write as a single fraction.

$$a \frac{2x}{5} + \frac{x}{5}$$

$$b \frac{2}{x} + \frac{1}{x}$$

$$c \frac{x}{7} + \frac{3x}{7}$$

$$d \frac{1}{7x} + \frac{3}{7x}$$

$$e \frac{5x}{8} + \frac{x}{4}$$

$$f \frac{5}{8x} + \frac{1}{4x}$$

$$g \frac{2x}{3} + \frac{x}{6}$$

$$h \frac{2}{3x} + \frac{1}{6x}$$

2 Sort into four pairs of equivalent fractions.

$$A \frac{x}{2} + \frac{x}{4}$$

$$B \frac{2}{x} + \frac{2}{x}$$

$$C \frac{9x}{8} - \frac{x}{4}$$

$$D \frac{7x}{8}$$

$$E \frac{5x}{x^2} - \frac{2}{x}$$

$$F \frac{3x}{4}$$

$$G \frac{3}{x}$$

$$H \frac{4}{x}$$

3 Simplify.

$$a \frac{3x}{4} + \frac{2x}{5}$$

$$b \frac{3}{4x} + \frac{2}{5x}$$

$$c \frac{3x}{4} - \frac{2x}{3}$$

$$d \frac{3}{4x} - \frac{2}{3x}$$

$$e \frac{x}{2} + \frac{x+1}{3}$$

$$f \frac{x-1}{3} + \frac{x+2}{4}$$

4 Work out these subtractions.

a $\frac{x+1}{3} - \frac{(2x+1)}{4}$

b $\frac{x-3}{3} - \frac{(x-2)}{5}$

c $\frac{(x+2)}{2} - \frac{(2x+1)}{7}$

5 Write as a single fraction.

a $\frac{2x-1}{5} + \frac{x+3}{2}$

b $\frac{1}{x} + \frac{2}{x+1}$

c $\frac{3}{x-2} + \frac{4}{x}$

d $\frac{5}{x-2} + \frac{3}{x+3}$

e $\frac{7}{x+1} - \frac{3}{x+2}$

f $\frac{x}{2} + \frac{x+3}{x+2}$

Multiplication and division

The method is similar to that used when multiplying and dividing ordinary numerical fractions.

Example 3

a $x^2 \div \frac{x}{2} = x^2 \times \frac{2}{x} = 2x$

b $\frac{2a}{3x} \times \frac{2x^2}{5a} = \frac{4\cancel{a}x^2}{15\cancel{a}x} = \frac{4x}{15}$

c $\frac{(x-1)}{(x+2)} \div \frac{2(x-1)}{(x+3)} = \frac{(x\cancel{-1})}{(x+2)} \times \frac{(x+3)}{2(x\cancel{-1})}$
 $= \frac{(x+3)}{2(x+2)}$

EXERCISE 3

Simplify the following:

1 $\frac{3x}{2} \times \frac{2a}{3x}$

2 $\frac{5mn}{3} \times \frac{2}{n}$

3 $\frac{3y^2}{x^2} \times \frac{2x}{9y}$

4 $\frac{5ab^2}{2} \times \frac{3}{2a^2b}$

5 $\frac{2}{a} \div \frac{a}{2}$

6 $\frac{4x}{3} \div \frac{x}{2}$

7 $\frac{x-1}{2} \div \frac{x+2}{2}$

8 $\frac{x}{5} \times \frac{y^2}{x^2}$

9 $\frac{x-1}{4} \times \frac{x}{2x-2}$

10 $\frac{x^2-4}{3} \times \frac{9}{x-2}$

Questions 11 onwards involve addition, subtraction, multiplication and division.

11 Copy and complete.

a $\frac{x^2}{3} \times \frac{\square}{x} = 2x$

b $\frac{8}{x} \div \frac{2}{x} = \square$

c $\frac{a}{3} + \frac{a}{3} = \frac{\square}{3}$

12 Sort these into four pairs of equivalent expressions.

A $\frac{x^2}{3x}$

B $\frac{x}{2} \times \frac{x}{2}$

C $\frac{12x+6}{6}$

D $\frac{x}{5} - \frac{2}{5}$

E $\frac{2x^2+x}{x}$

F $\frac{x(x+1)}{3x+3}$

G $\frac{x-2}{5}$

H $\frac{ax^2}{4a}$

13 Write as a single fraction in its simplest form.

a $\frac{3x}{2} \times \frac{2a}{3x}$

b $\frac{5mn}{3} \times \frac{2}{n}$

c $\frac{3y^2}{3} \times \frac{2x}{9y}$

d $\frac{2}{q} \div \frac{a}{2}$

e $\frac{4x}{3} \div \frac{x}{2}$

f $\frac{x}{5} \times \frac{y^2}{x^2}$

g $\frac{a^2}{5} \div \frac{a}{10}$

h $\frac{x^2}{x^2 + 2x} \div \frac{x}{x + 2}$

14 Simplify the following.

a $\frac{x}{y} \times \frac{xy}{z} \times \frac{z}{x^2}$

b $\frac{ab}{x} \times \frac{xb^2}{a^2} \times \frac{a^2}{x}$

c $\frac{x^2 + 7x}{x^2 - 1} \times \frac{x + 1}{x + 7}$

d $\left(\frac{x}{y}\right) \times \frac{z^3}{x}$

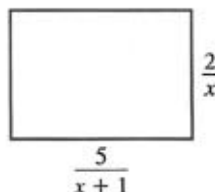
15 Copy and complete.

a $\frac{x}{5} - \frac{\square}{\square} = \frac{x}{10}$

b $\frac{\square}{2} + \frac{x}{4} = \frac{7x}{4}$

c $\frac{3}{2x} - \frac{1}{8x} = \frac{\square}{8x}$

16 The perimeter of the rectangle shown is 24 units.
Form an equation and solve it to find x .



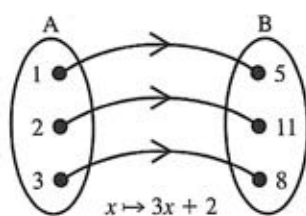
17 A rectangle measuring $\frac{3}{x}$ by $\frac{6}{x+1}$ has an area of 75 square units.
Find x .

1.2 Functions and mappings

If A and B are non-empty sets then a *mapping* from A to B is a rule which associates an element of B with every element of A.

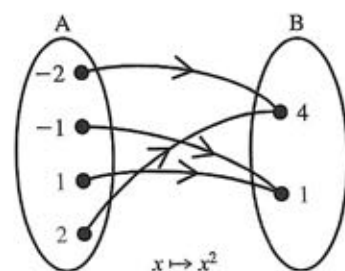
Here are two types of mappings.

Type 1



This is called a **one-to-one mapping**.

Type 2



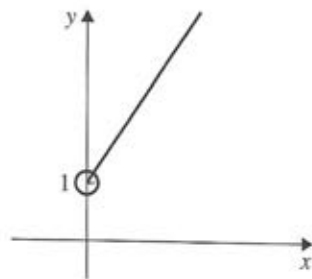
This is called a **two-to-one mapping** or a **many-to-one mapping**.

b $f(x) = 2x + 1$, for $x \in \mathbb{R}, x > 0$

The domain is all values of x greater than 0.

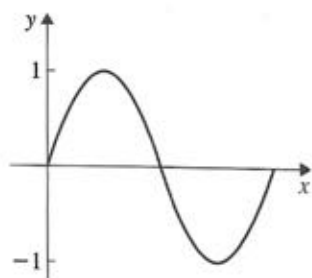
From the graph of $f(x) = 2x + 1$, we see that the range is $f(x) > 1$.

Notice that since $x = 0$ is not in the domain, the range is $f(x) > 1$ and *not* $f(x) \geq 1$.



c $f(x) = \sin x$, for $0^\circ \leq x \leq 360^\circ$

The range of the function is $-1 \leq f(x) \leq 1$.



EXERCISE 4

1 Decide which of these functions are one-to-one and which are many-to-one.

a $f: x \mapsto 2x - 5, x \in \mathbb{R}$

b $f: x \mapsto x^2 + 2, x \in \mathbb{R}$

c $f: x \mapsto \frac{1}{x}, x \in \mathbb{R}, x \neq 0$

d $f: x \mapsto x^3, x \in \mathbb{R}$

e $f: x \mapsto x(x - 1), x \in \mathbb{R}$

f $f: x \mapsto \cos x, 0^\circ \leq x \leq 360^\circ$

g $f: x \mapsto (x - 1)^2 + 2, x \in \mathbb{R}$

h $f: x \mapsto x(x - 1)(x - 2), x \in \mathbb{R}$

i $f: x \mapsto x^3 + 10, x \in \mathbb{R}$

j $f: x \mapsto \sqrt{x}, x \in \mathbb{R}, x \geq 0$

2 Find the range of each function.

a $f(x) = 2x + 1, x \in \mathbb{R}, x \geq 1$

b $f(x) = x^2 + 2, x \in \mathbb{R}$

c $f(x) = x + 1, x \in \mathbb{R}, x \geq 0$

d $f(x) = x^3, x \in \mathbb{R}, x \geq 0$

e $f(x) = \sin x, 0^\circ \leq x \leq 180^\circ$

f $f(x) = x(x - 3), 0 \leq x \leq 3$

g $f(x) = 5x - x^2, 0 \leq x \leq 5$

h $f(x) = \sqrt{x}, 0 \leq x \leq 25$

i $f(x) = \frac{1}{x}, x \geq 1$

j $f(x) = \frac{1}{1 + x^2}, x \in \mathbb{R}$

3 a By completing the square, write $f(x) = x^2 + 6x + 4$ in the form $(x + a)^2 + b$.

b Sketch the graph of $y = f(x)$ and state its range for $x \in \mathbb{R}$.

4 Find the range for each of these functions for $x \in \mathbb{R}$.

a $x^2 - 6x + 12$

b $x^2 + 20x + 50$

c $\frac{1}{x^2 + 2}$

d $\frac{1}{x^2 + 4x + 5}$

Composite functions

The function $f: x \mapsto 3x + 2$ is itself a composite function, consisting of two simpler functions: 'multiply by 3' and 'add 2'.

We can check the answer by substituting any value for x , say 3.

$$f(3) = \frac{3+1}{3} = \frac{4}{3}$$

$$f^{-1}\left(\frac{4}{3}\right) = \frac{1}{\frac{4}{3}-1} = \frac{1}{\frac{1}{3}} = 3 \text{ (as expected)}$$

EXERCISE 5

In this exercise the domain for each function is the set of real numbers unless otherwise stated.

For questions 1 and 2, the functions f , g and h are as follows:

$$f: x \mapsto 4x \quad g: x \mapsto x+5 \quad h: x \mapsto x^2$$

1 Find the following in the form ' $x \mapsto \dots$ '

- | | | | |
|--------|---------|---------|--------|
| a fg | b gf | c hf | d fh |
| e gh | f fgh | g hfg | |

2 Find

- | | |
|-------------------------|--------------------------|
| a x if $hg(x) = h(x)$ | b x if $fh(x) = gh(x)$ |
|-------------------------|--------------------------|

For questions 3, 4 and 5, the functions f , g and h are as follows:

$$f: x \mapsto 2x \quad g: x \mapsto x-3 \quad h: x \mapsto x^2$$

3 Find the following in the form ' $x \mapsto \dots$ '

- | | | |
|--------|---------|---------|
| a fg | b gf | c gh |
| d hf | e ghf | f hgf |

4 Evaluate

- | | | |
|------------|-------------|-------------|
| a $fg(4)$ | b $gf(7)$ | c $gh(-3)$ |
| d $fgf(2)$ | e $ggg(10)$ | f $hfh(-2)$ |

5 Find

- | | |
|------------------------|--------------------------|
| a x if $f(x) = g(x)$ | b x if $hg(x) = gh(x)$ |
| c x if $gf(x) = 0$ | d x if $fg(x) = 4$ |

For questions 6, 7 and 8, the functions l , m and n are as follows:

$$l: x \mapsto 2x+1 \quad m: x \mapsto 3x-1 \quad n: x \mapsto x^2$$

6 Find the following in the form ' $x \mapsto \dots$ '

- | | | |
|--------|---------|---------|
| a lm | b ml | c ln |
| d nm | e lnm | f mln |

7 Find

- | | | |
|-----------|------------|------------|
| a $lm(2)$ | b $nl(1)$ | c $mn(-2)$ |
| d $mm(2)$ | e $nln(2)$ | f $llm(0)$ |

8 Find

- a x if $l(x) = m(x)$
- b two values of x if $nl(x) = nm(x)$
- c x if $ln(x) = mn(x)$

In questions 9 to 20, find the inverse of each function in the form ' $x \mapsto \dots$ '

9 $f: x \mapsto 5x - 2$

10 $f: x \mapsto 5(x - 2)$

11 $f: x \mapsto 3(2x + 4)$

12 $g: x \mapsto \frac{2x + 1}{3}$

13 $f: x \mapsto \frac{3(x - 1)}{4}$

14 $g: x \mapsto 2(3x + 4) - 6$

15 $h: x \mapsto \frac{1}{2}(4 + 5x) + 10$

16 $k: x \mapsto -7x + 3$

17 $j: x \mapsto \frac{12 - 5x}{3}$

18 $l: x \mapsto \frac{4 - x}{3} + 2$

19 $m: x \mapsto \frac{\left[\frac{(2x - 1)}{4} - 3\right]}{5}$

20 $n: x \mapsto \frac{12}{x}, x \neq 0$

For questions 21 to 24, the functions f , g and h are defined as follows:

$f: x \mapsto 3x$ $g: x \mapsto x - 5$ $h: x \mapsto 2x + 1$

21 Find the form ' $x \mapsto \dots$ '

- a f^{-1}
- b g^{-1}
- c h^{-1}
- d fg
- e $(fg)^{-1}$
- f $g^{-1}f^{-1}$

22 Find the form ' $x \mapsto \dots$ '

- a hf
- b hf^{-1}
- c $f^{-1}h^{-1}$
- d hg
- e $(hg)^{-1}$
- f $g^{-1}h^{-1}$

23 Find

- a $g^{-1}(2)$
- b $fg^{-1}(2)$
- c $(gf)^{-1}(10)$
- d $f^{-1}g^{-1}(10)$
- e $f^{-1}f^{-1}ff(2)$

24 Find

- a x if $h(x) = f(x)$
- b the set of values of x for which
 - i $f(x) > g(x)$
 - ii $fg(x) > 0$

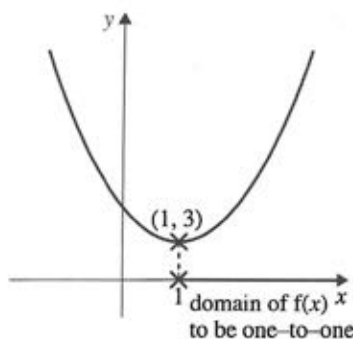
25 The function f is defined by $f: x \mapsto \frac{2x + 5}{3}$

- a Find f^{-1} in the form ' $x \mapsto \dots$ '
- b Find $f^{-1}(3)$
- c Show that $f^{-1}(3)$ is the solution of the equation $f(x) = 3$.

- a Sketch the curve $y = (x - 1)^2 + 3$

We see that $f(x)$ is a one-to-one function for $x \geq 1$

$$\therefore p = 1$$



- b $y = (x - 1)^2 + 3$

Rearrange to make x the subject,

$$\begin{aligned}(x - 1)^2 &= y - 3 \\ x - 1 &= \pm \sqrt{y - 3} \\ x &= 1 \pm \sqrt{y - 3}\end{aligned}$$

We require the positive square root, $x = 1 + \sqrt{y - 3}$

$$\therefore f^{-1}(x) = 1 + \sqrt{x - 3}$$

The domain of $f^{-1}(x)$ is $x \geq 3$

Notice that the range of $f(x)$ is $f(x) \geq 3$.

EXERCISE 6

- 1 Find the inverse of the following functions

a $f(x) = 2 - x, \quad x \in \mathbb{R}$

b $f(x) = \frac{13}{2x}, \quad x \in \mathbb{R}, x \neq 0$

c $f(x) = 3 - 5x, \quad x \in \mathbb{R}$

d $f(x) = \frac{2x + 5}{x - 7}, \quad x \in \mathbb{R}, x \neq 7$

e $f(x) = \frac{3x - 5}{2x + 7}, \quad x \in \mathbb{R}, x \neq -\frac{7}{2}$

- 2 The function $f(x)$ is defined by $f(x) = x^2 + 1, x \geq 0$. On the same diagram draw a sketch of $y = f(x)$ and $y = f^{-1}(x)$.

- 3 a If $f(x) = 3x - 1$ and $g(x) = x + 7$, calculate $f^{-1}g(x)$ and $gf^{-1}(x)$.

- b Find the values of $f^{-1}g(4)$ and $gf^{-1}(2)$.

- 4 A function is defined by $f(x) = 3x - 1$ for all x .

- a Find $ff(x)$ and $f^{-1}(x)$.

- b Show that $(ff)^{-1}(x)$ is identically equal to $f^{-1}f^{-1}(x)$.

- 5 A function is defined by $f(x) = \frac{x + 10}{x - 8}, x \neq 8$.

- a Find $f^{-1}(x)$.

- b Find $f^{-1}(5)$.

- c Find a positive integer p such that $f(p) = p$.

- d Find also $f^{-1}(p)$ for the positive integer found in part c.

- 6 a Sketch the curve of $y = x^2 - 2x - 3$ showing where the curve cuts the x -axis.
 b You are given the function $f(x) = x^2 - 2x - 3, x \geq 1$.
 i Find the range of f .
 ii State the range and domain of f^{-1} .
 iii Sketch the graph of f^{-1} , showing where the graph meets the coordinate axes.
- 7 A function f is defined by $f: x \mapsto 2 - \frac{1}{x}, x \neq 0$.
 a Find f^{-1} and state the value of x for which f^{-1} is not defined.
 b Find the value of x for which $f(x) = f^{-1}(x)$.
- 8 The function f is defined by $f: x \mapsto (x - 2)^2, x \geq 2$.
 i State the domain and range of the inverse function f^{-1} .
 ii Find an expression for $f^{-1}(x)$.
 iii Find a solution to the equation $f(x) = f^{-1}(x)$.
- 9 The one-to-one functions f and g are defined by $f(x) = \sqrt{x - 2}$ for $x \geq a$ and $g(x) = \frac{1}{x^2}$ for $x > b$.
 a Find the smallest possible values of a and b .
 b Find $f^{-1}(x)$ and $g^{-1}(x)$ and state the restrictions, if any, on the domains of these inverse functions.
- 10 The functions f and g are defined by $f(x) = 2x + 1$ for all x and $g(x) = \frac{5}{x - 3}, x \neq 3$. Find $f^{-1}(x)$ and $g^{-1}(x)$ and state the restrictions, if any, on the domains of these inverse functions.
- 11 Copy and complete the domains for the following functions so that the domains are as large as possible for the functions to be one-to-one:
- | | |
|--------------------------|----------------------------|
| a $f(x) = 3x - 1$ | $\dots \leq x \leq \dots$ |
| b $f(x) = 2x^2$ | $x \geq \dots$ |
| c $f(x) = \sin x$ | $-90^\circ < x \leq \dots$ |
| d $f(x) = (x - 1)^2 + 3$ | $x \geq \dots$ |
| e $f(x) = x^2 + 2x - 3$ | $x \leq \dots$ |
| f $f(x) = x^2 + 8x + 13$ | $x \geq \dots$ |

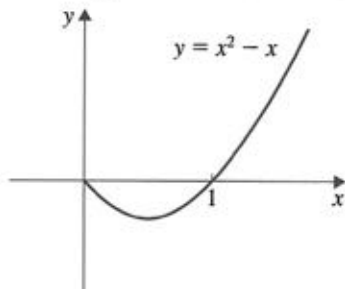
1.3 The modulus function

The **modulus** of x is written $|x|$ and means 'the positive value of x '.
 For example, $|-2| = 2, |7| = 7$

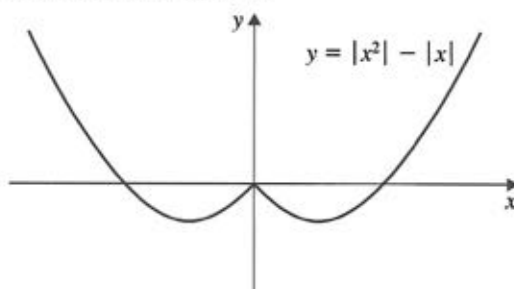
Example 7

Sketch the graph of $y = |x|^2 - |x|$.

a Sketch $y = x^2 - x$ for $x \geq 0$



b Reflect in the y axis



EXERCISE 7

1 Sketch the graphs of the following, marking where the graphs cross the axes.

a $y = |x - 4|$

b $y = |x + 2|$

c $y = |3x - 1|$

d $y = |x^2 - 2x|$

e $y = |x^2 - 9|$

f $y = |x^2 - 5x + 4|$

g $y = \left| \frac{1}{x} \right|$

h $y = |x^3 - x^2 - 6x|$

2 Solve the following equations.

a $|x - 2| = 5$

b $|2x - 3| = 3$

c $|1 - 2x| = 5$

d $|x^2 - 5| = 4$

e $|x^2 - 37| = 12$

f $2|x + 3| = 8$

g $3|2x - 1| = 9$

h $\left| \frac{1}{x} \right| = 2$

3 Solve the following inequalities.

a $|x - 4| > 5$

b $|x + 1| < 3$

c $|2x + 3| > 7$

d $|3x - 1| > -1$

e $|x + 2| < \frac{1}{2}x + 5$

f $|3x - 2| < 6 - x$

g $|x| > |2x - 1|$

h $|x + 4| > |x - 2|$

i $2|x - 1| < |x - 2|$

4 Sketch the graphs of $y = |3x - 1|$ and $y = |x^2 - 5x + 4|$ and hence solve the equation $|3x - 1| = |x^2 - 5x + 4|$.

5 Sketch the graphs of the following, marking where the graph crosses the axes.

a $y = |x| - 2$

b $y = |x| + 3$

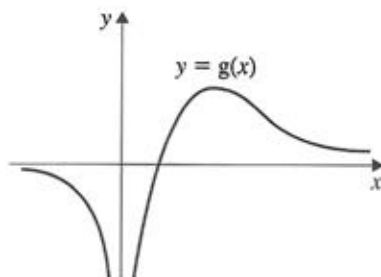
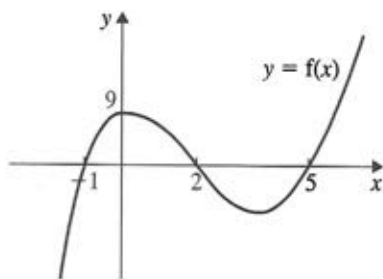
c $y = 2|x| - 1$

d $y = |x|^2 - 2|x|$

e $y = \cos |x|$

f $y = \frac{1}{|x|}$

- 6 The graphs of $y = f(x)$ and $y = g(x)$ are shown below.



Sketch the graphs of

- a $y = |f(x)|$ b $y = f(|x|)$ c $y = |g(x)|$ d $y = g(|x|)$

- 7 a Sketch the graph of $y = \sin |x|$.
b How many solutions are there to the equation

$$\sin |x| = \frac{1}{2}, \text{ for } -180^\circ \leq x \leq 180^\circ?$$

- 8 a Sketch the graphs of $f(x) = |x| - 3$ and $g(x) = \frac{1}{3}x$.
b Hence solve the inequality $|x| - 3 > \frac{1}{3}x$.

- 9 a On the same diagram, sketch the graphs of

$$y = \frac{1}{x-1} \text{ and } y = 9|x-1|.$$

- b Hence find the set of values of x for which

$$\frac{1}{x-1} < 9|x-1|.$$

- 10 a Sketch the graphs of the following, marking where the graphs cut the axes.

a i $y = 2 - |x + 1|$

ii $y = 5 - |x - 1|$

- b State the range of each function.

- 11 Given that $f(x) = x^2 - 3x$, $x \in \mathbb{R}$, sketch in separate diagrams the graphs of

a $y = f(x)$

b $y = |f(x)|$

c $y = f(|x|)$

1.4 Transformations of the graph of a function

In 'Pure Mathematics C1 C2' we discussed the important single transformations of the graph of a function.

Here is a summary:

$y = f(x) + a$ Translation by a units parallel to the y -axis.

$y = f(x - a)$ Translation by a units parallel to the x -axis (note the negative sign).

$y = af(x)$ Stretch parallel to the y -axis by a scale factor a .

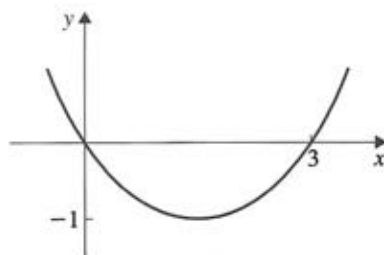
EXERCISE 8

- 1 The graph of $y = f(x)$ is shown.

Sketch the graphs of,

a $y = f(x + 2)$

b $y = -f(x)$



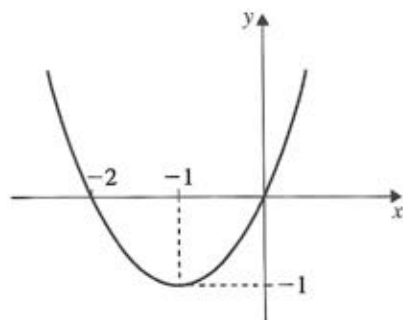
- 2 This is the sketch graph of $y = f(x)$.

a Sketch the graph of $y = f(x) + 2$

b Sketch the graph of $y = f(x - 1)$

c Sketch the graph of $y = -f(x)$

d Sketch the graph of $y = f(-x)$.



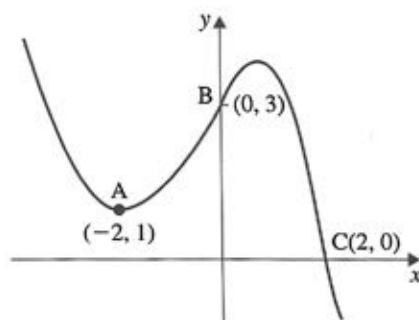
- 3 This is the sketch of $y = f(x)$ which passes through A, B, C.

Sketch the following curves, giving the new coordinates of A, B, C in each case.

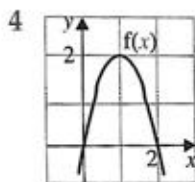
a $y = 2f(x)$

b $y = f(x - 2)$

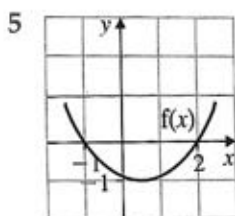
c $y = f(\frac{1}{2}x)$



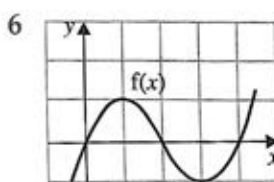
In questions 4 to 9 each graph shows a different function $f(x)$. On squared paper draw a sketch to show the given transformation. The scales are 1 square = 1 unit on both axes.



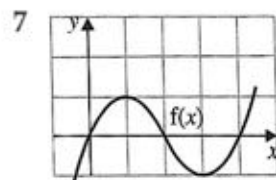
Sketch $f(x + 2)$



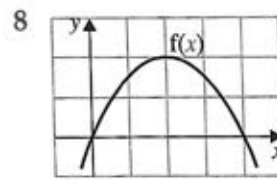
Sketch $f(-x)$



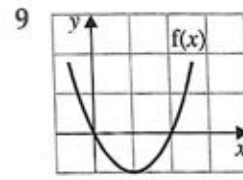
Sketch $f(\frac{1}{2}x)$



Sketch $2f(x)$



Sketch $f(2x)$

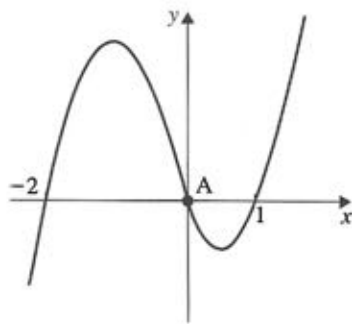


Sketch $-f(x)$

- 10 This is the sketch graph of $y = f(x)$.

- a Sketch $y = f(x - 3)$
b Sketch $y = f(x - 3) + 4$

Give the new coordinates of the point A on the two sketches.



- 11 On the same axes, sketch and label:

- a $y = x^2$
b $y = (x - 2)^2$
c $y = (x - 2)^2 + 4$

- 12 a Sketch and label $y = f(x)$, where $f(x) = x(x - 4)$.

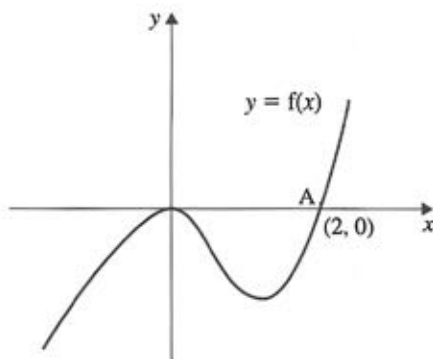
- b On the same axes, sketch and label:

- i $y = f(2x)$ ii $y = -f(2x)$

- 13 This is a sketch of $y = f(x)$.

- a Sketch $y = f(x + 1) + 5$
b Sketch $y = f(x - 3) - 4$
c Sketch $y = 1 + f(2x)$.

Show the new coordinates of the point A on each sketch.



- 14 a Find the equation of the curve when $y = x^3$ is translated 5 units in the direction $\uparrow$.

- b Find the equation of the curve when $y = x^3$ is translated 2 units in the direction $\rightarrow$.

- c Find the equation of the curve when $y = x^3$ is translated by the vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

- 15 a Sketch the graph of $y = \sin x$, for $0 \leq x \leq 360^\circ$.

- b Sketch the graph of $y = \sin \frac{1}{2}x$, for $0 \leq x \leq 360^\circ$.

- 16 a Sketch the graph of $y = \tan x$, for $0 \leq x \leq 2\pi$.

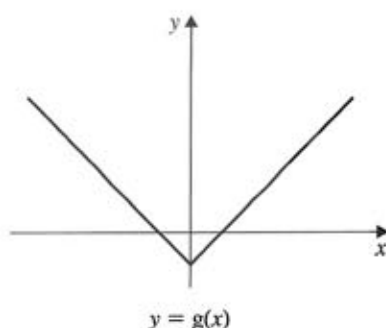
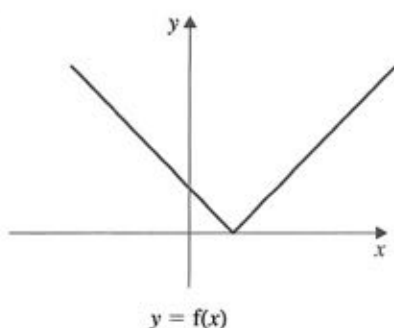
- b Sketch the graph of $y = \tan\left(x - \frac{\pi}{2}\right)$, for $0 \leq x \leq 2\pi$.

- 17 You are given $f(x) = 2x + 1$ and $g(x) = 6x - 1$.

Find a pair of successive transformations which, applied to $f(x)$, will give $g(x)$.

- 18 You are given $f(x) = \frac{1}{2}x - 2$ and $g(x) = 2x + 1$.

Find a pair of successive transformations which, applied to $f(x)$, will give $g(x)$.



The diagrams show the graphs of $y = f(x)$ and $y = g(x)$, where

$$f(x) = |x - 4| \text{ and } g(x) = |x| - 3, x \in \mathbb{R}$$

Describe the geometrical transformations by which each of the above graphs can be obtained from the graph of $y = |x|$.

EXAMINATION EXERCISE 1

1 Simplify $\frac{x^3 - 3x^2}{x^3 - 9x}$. [OCR]

2 Express $\frac{x}{(x+1)(x+3)} + \frac{x+12}{x^2-9}$ as a single fraction in its simplest form. [EDEXCEL]

3 Express $\frac{3}{x^2+2x} + \frac{x-4}{x^2-4}$ as a single fraction in its simplest form. [EDEXCEL]

4 The function f is defined by

$$f: x \mapsto \frac{1}{\sqrt{x}} + 2, \quad x > 0.$$

i State the range of f .

ii Find an expression for $f^{-1}(x)$. [OCR]

5 The functions f and g are defined with their respective domains by

$$f: x \mapsto \frac{4}{3+x}, \quad x > 0$$

$$g: x \mapsto 9 - 2x^2, \quad x \in \mathbb{R}$$

a Find $fg(x)$, giving your answer in its simplest form.

b i Sketch the graph of $y = g(x)$.

ii Find the range of g .

c i Solve the equation $g(x) = 1$.

ii Explain why the function g does not have an inverse.

d The inverse of f is f^{-1} .

i Find $f^{-1}(x)$.

ii Solve the equation $f^{-1}(x) = f(x)$. [AQA]

6 $f(x) = \frac{2}{x-1} - \frac{6}{(x-1)(2x+1)}, x > 1.$

a Prove that $f(x) = \frac{4}{2x+1}.$

b Find the range of $f.$

c Find $f^{-1}(x).$

d Find the range of $f^{-1}(x).$

[EDEXCEL]

7 The function f is given by

$$f: x \mapsto \frac{x}{x^2-1} - \frac{1}{x+1}, x > 1.$$

a Show that $f(x) = \frac{1}{(x-1)(x+1)}.$

b Find the range of $f.$

The function g is given by

$$g: x \mapsto \frac{2}{x}, x > 0.$$

c Solve $gf(x) = 70$

[EDEXCEL]

8 The functions f and g are defined for all real values of x by

$$f(x) = 3x - 10,$$

$$g(x) = |x - 2|.$$

i State the range of each function.

ii Show that the value of $fg(-4)$ is 8.

iii Determine the value of $f^{-1}(7).$

iv Solve the equation $gf(x) = 1.$

[OCR]

9 The functions $f(x)$ and $g(x)$ are defined by

$$f(x) = \ln x, g(x) = x^2.$$

i Write down expressions for $fg(x)$ and $gf(x).$

ii Show that $fg(x) = kf(x)$, for some constant k , and find $k.$

[MEI]

10 a Sketch, on the same diagram, the graphs of

$$y = |2x + 3| \text{ and } y = 2x^2 - 9,$$

stating the coordinates of any points where the graphs meet the axes.

b Deduce the number of roots of the equation

$$|2x + 3| = 2x^2 - 9.$$

Determine the value of each of these roots.

[AQA]

11

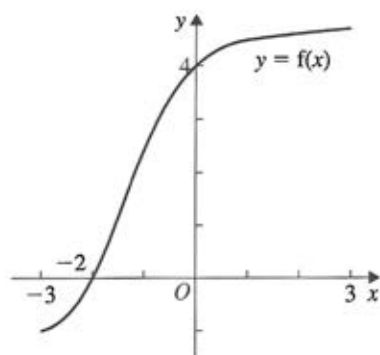


Fig. 1

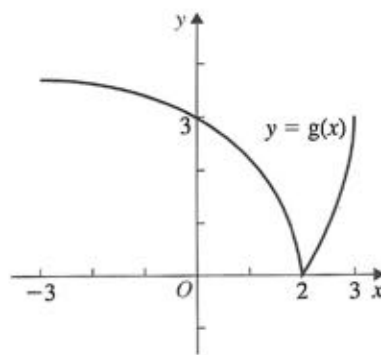


Fig. 2

Each of the functions f and g has domain $-3 \leq x \leq 3$. Fig. 1 shows the curve $y = f(x)$, which meets the x -axis at $(-2, 0)$ and the y -axis at $(0, 4)$. Fig. 2 shows the curve $y = g(x)$, which meets the x -axis at $(2, 0)$ and the y -axis at $(0, 3)$.

- Determine the value of $gf(-2)$.
- State which one of the functions f and g is one-one and sketch the graph of its inverse. You should indicate on your sketch the coordinates of intersections with the axes.
- Given that $g(x) = |h(x)|$, where $g(x) \neq h(x)$, sketch the graph of a possible curve $y = h(x)$. [OCR]

12 The functions f and g are defined by

$$f: x \mapsto x^2 - 2x + 3, x \in \mathbb{R}, 0 \leq x \leq 4,$$

$$g: x \mapsto \lambda x^2 + 1, \text{ where } \lambda \text{ is a constant, } x \in \mathbb{R}.$$

- Find the range of f .
- Given that $gf(2) = 16$, find the value of λ . [EDEXCEL]

13 The function f has domain $x \geq 4$ and is defined by $f(x) = (x - 3)^2 + 1$.

- Find the value of $f(4)$ and sketch the graph of $y = f(x)$.
 - Hence find the range of f .
- Explain why the equation $f(x) = 1$ has no solution.
- The inverse function of f is f^{-1} . Find $f^{-1}(x)$. [AQA]

14 Figure 3 shows a sketch of the curve with equation $y = f(x)$, $-1 \leq x \leq 3$. The curve touches the x -axis at the origin O , crosses the x -axis at the point $A(2, 0)$ and has a maximum at the point $B(\frac{4}{3}, 1)$.

In separate diagrams, show a sketch of the curve with equation

- $y = f(x + 1)$,
- $y = |f(x)|$,
- $y = f(|x|)$,

marking on each sketch the coordinates of points at which the curve

- has a turning point,
- meets the x -axis.

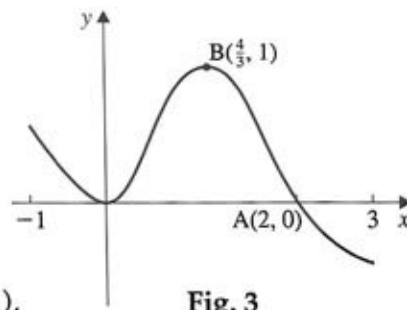


Fig. 3

[EDEXCEL]

- 15 Sketch the graph of the function $y = |2x - 1|$, and state the coordinates of any points where the graph meets the axes. [MEI]

16 Solve the equation $|2x + 1| = 6$. [MEI]

17 Solve the inequality $|3x - 2| > 4$. [MEI]

- 18 The functions f and g are defined as follows:

$$f: x \mapsto 5e^{\frac{1}{2}x} - 6, x \in \mathbb{R},$$

$$g: x \mapsto |x - 2|, x \in \mathbb{R}.$$

- i Determine the range of f and the range of g .
- ii Find an expression for $f^{-1}(x)$.
- iii Find the solutions of the equation $gf(x) = 7$, giving each in an exact form. [OCR]

- 19 The function f is defined for all real values of x by

$$f(x) = |2x - 3| - 1.$$

- a Sketch the graph of $y = f(x)$. Indicate the coordinates of the points where the graph crosses the x -axis and the coordinates of the point where the graph crosses the y -axis.
- b State the range of f .
- c Find the values of x for which $f(x) = x$. [AQA]

- 20 Find the exact solution of the equation

$$|7x - 3| = |7x + 6|. \quad \text{[OCR]}$$

- 21 Describe, in each of the following cases, a single transformation which maps the graph of $y = e^x$ onto the graph of the function given.

a $y = e^{3x}$.

b $y = e^{x-3}$.

c $y = \ln x$. [AQA]

- 22 i Given that $f(x) = x^2$, sketch the graph of $y = f(x)$.

The graph of $y = g(x)$ is obtained by reflecting the graph of $y = f(x)$ in the x -axis. The graph of $y = h(x)$ is obtained by translating the graph of $y = g(x)$ by +2 units parallel to the y -axis.

- ii Sketch and label the graphs of $y = g(x)$ and $y = h(x)$ on a single diagram.
- iii Write down expressions for $g(x)$ and $h(x)$ in terms of x . [OCR]

23 The diagram shows the graphs of $y = x$ and $y = f(x)$.

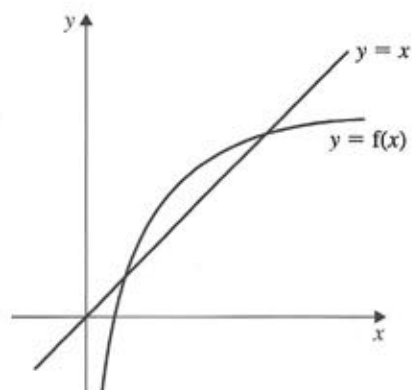
a i Describe the geometrical transformation by which the graph of $y = f^{-1}(x)$ can be obtained from the graph of $y = f(x)$.

ii Copy the diagram and sketch on the same axes the graph of $y = f^{-1}(x)$.

b The function f is defined for $x > 0$ by $f(x) = 3 \ln x$.

i Describe the geometrical transformation by which the graph of $y = f(x)$ can be obtained from the graph of $y = \ln x$.

ii Find an expression for $f^{-1}(x)$.



[AQA]